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Mykola BEREZKYI
West Ukrainian National University

ANALYSIS OF FRESCHET AND HAUSDORF METRICS AND THEIR MODIFICATIONS FOR IMAGE COMPARISON

This paper provides a comprehensive analysis of classical and modern metric approaches used for quantitative evaluation of image similarity, including the Fréchet and Hausdorff distances as well as their modifications – the Gromov-Fréchet and Gromov-Hausdorff metrics. The relevance of this research is determined by the wide use of image comparison methods in computer vision systems, where they form the basis for segmentation, classification, and object detection in various application domains, particularly in medicine. Images are represented as polygons, which unifies computational procedures and simplifies the formal description of distance measurement algorithms.

The properties of the considered metrics were compared, and computational experiments demonstrated that the Fréchet distance effectively reflects the similarity of polygon contours, while the Hausdorff distance is more suitable for comparing inner regions. The Gromov-based modifications provide minimal distances and more flexible results when dealing with objects of complex structure. Algorithmic solutions for each metric are described, with an emphasis on their computational complexity and possible practical applications. Special attention is given to isometric transformations, which reduce matching errors.

The results were validated through experiments implemented in Java with the OpenCV library, proving the adequacy and efficiency of the proposed approaches. The practical value of the research lies in the potential integration of the obtained results into automated medical diagnostic systems for the analysis of histological, cytological, and immunohistochemical images. The proposed algorithms may serve as a basis for developing effective segmentation and classification methods for biomedical data.

Keywords: metric, Hausdorff distance, Fréchet distance, Gromov-Hausdorff metric, Gromov-Fréchet metric, images, polygons.

Микола БЕРЕЗЬКИЙ
Західноукраїнський національний університет

АНАЛІЗ МЕТРИК ФРЕШЕ ТА ХАУСДОРФА Ї ЇХ МОДИФІКАЦІЙ ДЛЯ ПОРІВНЯННЯ ЗОБРАЖЕНЬ

У статті здійснено комплексний аналіз класичних і сучасних метричних підходів, що застосовуються для кількісної оцінки подібності зображень, зокрема метрик Фреше, Хаусдорфа, а також їх модифікацій — відстаней Громова-Фреше та Громова-Хаусдорфа. Показано, що актуальність цих досліджень визначається широким використанням методів порівняння зображень у системах комп'ютерного зору, де вони є основою для сегментації, класифікації та виявлення об'єктів у різних прикладних галузях, включаючи медицину. У роботі запропоновано розглядати зображення у вигляді полігонів, що уніфікує алгоритми обчислення та спрощує формальний опис процедур вимірювання відстаней.

Здійснено порівняння властивостей метрик та проведено обчислювальні експерименти, які продемонстрували, що відстань Фреше ефективно відображає подібність контурів полігонів, тоді як метрика Хаусдорфа є доцільною для порівняння внутрішніх областей. Модифікації на основі підходів Громова забезпечують знаходження мінімальних відстаней і надають більш гнучкі результати при роботі зі складними структурами. Для кожної метрики наведено алгоритмічні рішення, охарактеризовано їхню обчислювальну складність і можливості практичного застосування. Окрему увагу приділено методам ізометричних перетворень, які зменшують похибку при зіставленні об'єктів.

Результати підтверджені експериментами із застосуванням Java та бібліотеки OpenCV, що демонструють адекватність і продуктивність запропонованих підходів. Практична цінність дослідження полягає у можливості впровадження отриманих результатів у системи автоматизованого медичного діагностування для аналізу гістологічних, цитологічних та імуногістохімічних зображень. Запропоновані алгоритми можуть слугувати основою для створення ефективних методів сегментації та класифікації біомедичних даних.

Ключові слова: метрика, метрика Хаусдорфа, метрика Фреше, метрика Громова-Хаусдорфа, метрика Громова-Фреше, зображення, полігони.

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Introduction

In computer vision, image comparison is a relevant task. The operation of image comparison is used in image segmentation and classification. A specific image consists of an outer boundary, called a contour, and a set of inner points – the internal region.

To quantitatively evaluate the results of image comparison, metrics are applied [1]. The most well-known are the Fréchet metric and the Hausdorff metric. The Fréchet metric is used to compare curves [2–12].

In our case, the curves correspond to the contours of images. The Hausdorff metric is applied for comparing image regions. A number of studies have developed methods and algorithms for image comparison in terms of the Hausdorff metric [13–15].

Recently, modifications of the Fréchet and Hausdorff metrics have been proposed. First, the Gromov–Hausdorff metric was introduced, followed later by the Gromov–Fréchet metric.

At the West Ukrainian National University, a research group led by Professor O. M. Berezsky has been working for more than twenty years on the analysis of biomedical images. Biomedical images (cytological, histological, and immunohistochemical) are used for diagnosis in oncology. Some of the results are presented in the following research works [16–24].

In this article, for the sake of simplification, images are represented as polygons.

Problem Statement

Let two polygons *P* and *Q* be given.

We divide polygons *P* and *Q* into the outer contour and the internal region (fig. 1):

$$P = C_1 \cup O_1, \qquad Q = C_2 \cup O_2,$$

where C_1, O_1 – the contour and region of polygon *P*,

C_2, O_2 – the contour and region of polygon *Q*.

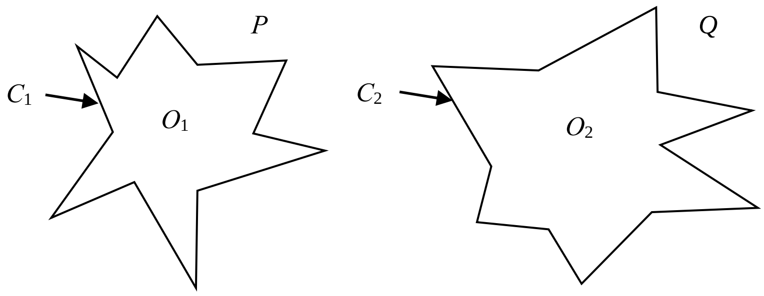


Fig. 1. Polygons P and Q

The distance between polygons *P* and *Q* is defined as the distance between their contours C_1 and C_2, and between their inner regions O_1 and O_2.

To determine the distance between polygons, it is necessary to compute the distance between both contours and regions. The Fréchet distance is used to evaluate the distance between contours, while the Hausdorff distance is applied to evaluate the distance between inner regions.

Polygonal contours are represented as closed or open polygonal curves (i.e., chains of points connected by straight-line segments). They can be divided into segments, which are sequences of line segments forming the curve.

We denote the contour of polygon *P* as $C_1 = (u_1, u_2, ..., u_p)$, where *p* is the total number of points on the curve C_1, and the contour of polygon *Q* as $C_2 = (v_1, v_2, ..., v_q)$, where *q* is the total number of points on the curve C_2.

To compute the distance between the contours, the discrete Fréchet distance is applied:

$$d_F(C_1, C_2) = \inf \left\{ \varepsilon_1 > 0 \mid \forall i = \overline{1, p}, \exists j = \overline{1, q} : d_F\left(C_1, C_2\right) \le \varepsilon_1 \text{ and conversely } \forall j = \overline{1, q},\ \exists i = \overline{1, p} : d_F\left(C_2, C_1\right) \le \varepsilon_1 \right\},$$

where *i* is the index of a point (vertex) on curve C_1;

j is the index of a point (vertex) on curve C_2;

ε_1 is the minimum admissible threshold distance value at which a mutual correspondence between all points of both curves is satisfied.

To find the minimal distance between two contours we use the Gromov–Fréchet distance:

$$d_{GF}(C_1, C_2) := \inf_{X, f, g} d_F^X(f(C_1), g(C_2)),$$

where $f : C_1 \to X$, $g : C_2 \to X$ are isometric embeddings into a metric space (*X*, *d*);

d_F^X is the usual Fréchet distance in the space *X*.

The distance between the internal regions of polygons is computed on the basis of the discrete Hausdorff distance:

$$d_H(O_1, O_2) = \inf \left\{ \varepsilon > 0 \mid \forall i = \overline{1, n}, \exists j = \overline{1, m} \text{ such that } d_H\left(O_1, O_2\right) \le \varepsilon_2 \text{ and conversely } \forall j = \overline{1, m},\ \exists i = \overline{1, n} \text{ such that } d_H\left(O_2, O_1\right) \le \varepsilon_2 \right\},$$

where *i* is the index of points of the internal region O_1;

j is the index of points of the internal region O_2;

n is the number of points covering the internal region O_1;

m is the number of points covering the internal region O_2;

ε_2 is the minimal admissible distance such that every point of one set (the internal region of one polygon) has a corresponding nearby point in the other set.

The minimal distance between the regions O_1 and O_2 is determined using the Gromov–Hausdorff distance:

$$d_{GH}(O_1,O_2) := \inf_{X,f,g} d_H^X(f(O_1),g(O_2)),$$

where $f: O_1 \to X$, $g: O_2 \to X$ are isometric embeddings into a metric space (X, d);

d_H^X is the usual Hausdorff distance in the space X.

For the given polygons, it is necessary to compute the following distances:

1. The discrete Hausdorff distance d_H.
2. The discrete Fréchet distance d_F.
3. The Gromov–Hausdorff distance d_{GH}.
4. The Gromov–Fréchet distance d_{GF}.

Presentation of the Main Material

To analyze the Fréchet and Hausdorff metrics and their modifications, we introduce the concept of a metric.

Let be an arbitrary set in a topological space [25]. Then a function $d: X \times X \to R$ is called a metric on X if, for all $x, y, z \in X$, the following conditions hold:

1) $d(x, y) \geq 0$ – non-negativity;
2) d (x, y) = 0 if and only if x = y (identity of indiscernibles);
3) $d(x, y) = d(y, x)$ – symmetry;
4) $d(x, y) \leq d(x, z) + d(z, y)$ – triangle inequality.

If conditions (2, 3, 4) are satisfied, the function is called a pseudometric. A quasi-metric is defined when only conditions (1, 2, 4) hold.

A pair (X, d), where d is a metric on the set X, is called a metric space.

In a metric space (X, d), for any $x \in X$ and $r > 0$, the set $B_r(x) = \{y \in X \mid d(x, y) < r\}$ is an open ball of radius r centered at x. If $A \subseteq X$, then $B_r(A) = \cup_{a\in A} B_r(a)$ is the r-neighborhood of the set A. A set X is bounded if it is contained within some ball.

A mapping $f: X \to Y$ from a metric space (X, d) into a metric space (Y, ρ) is an isometric embedding if $\rho(f(x), f(y)) = d(x, y)), \forall\ x, y \in X$.

A mapping of metric spaces is called an isometry if it is a bijection and preserves distances. Any isometric embedding is an isometry onto its image. The set of all isometries of a metric space onto itself forms a group under composition. A mapping of metric spaces is called non-expansive if it does not increase the distance between points.

Hausdorff Distance

For a metric space and two non-empty bounded subsets A, C of X, the Hausdorff distance between them is equal to:

$$d_H(A,C) = \inf\{r > 0 \mid A \subset B_r(C), C \subset B_r(A)\}.$$

Another description of the Hausdorff metric uses the notion of a correspondence. For two sets A and B, we define a subset $C \subset A \times B$ if the following conditions hold:

1. For every $a \in A$, there exists $b \in B$ such that $(a,b) \in C$.
2. For every $b \in B$, there exists $a \in A$ such that $(a,b) \in C$.

If C' is a correspondence between A_1 and A_2 and C' is a correspondence between A_2 and A_3, then the subset

$$C = C' \circ C'' = \{(a_1, a_3) \in A_1 \times A_3 \mid (a_1, a_2) \in C', (a_2, a_3) \in C'' \text{ for some } a_2 \in A_2\}$$

Then the Hausdorff metric d_H is defined by the formula [25]:

$$d_H(A,B) = \min\{\max\{d(a,b) \mid (a,b) \in C\} \mid C \text{ is a correspondence between } A \text{ and } B\}.$$

The Hausdorff metric on compact subsets of a complete metric space is complete.

For any convex sets with non-empty interior $A, B \subset \mathrm{R}^k$, the following holds:

$$d_H(A,B) = d_H(\partial A, \partial B),$$

where ∂ denotes the boundary of a set. If $A, B \subset \mathrm{R}^2$ are polygonal, non-empty and convex, then the computational complexity of finding $d_H(A,B)$ is $O(m+n)$, where m, n are the number of vertices of sets A and B, respectively [26]. The Hausdorff distance has many applications [27,28].

Gromov–Hausdorff Distance

Let two compact metric spaces (X_1, d_1), (X_2, d_2) be given. Then the Gromov–Hausdorff distance between them is defined as:

$$d_{GH}(X_1, X_2) = \inf\{d_H(j_1(X_1), j_2(X_2)) \mid j_i : X_i \to Z, i = 1,2 \text{ are isometric embeddings into a space } Z\}.$$

This is a correct definition of the distance d_{GH}, since there exists a metric space *Z* that includes isometric copies of X_1 and X_2. We select two base points $x_i \in X_i$, $i = \overline{1,2}$, and consider the bouquet $X_1 \vee X_2$, $X_1 \vee X_2 = (X_1 \cup X_2)/\{x_1, x_2\}$, where the metric *d* on $X_1 \vee X_2$ is defined by the following condition: *d* coincides with d_i on X_i, $i = \overline{1,2}$ and if $a_i \in X_i \subset X$, $i = \overline{1,2}$ then

$$d(a_1, a_2) = d_1(a_1, x_1) + d_2(a_2, x_2).$$

There also exists an equivalent definition of the Gromov–Hausdorff distance in terms of correspondences. A correspondence *C* between two non-empty sets *X* and *Y* is a subset $C \subset X \times Y$ such that:

1. For every $x \in X$, there exists $y \in Y$ with $(x, y) \in C$.

2. For every $y \in Y$, there exists $x \in X$ with $(x, y) \in C$.

The set $C(X, Y)$ is thus a relation that connects all points of *X* and *Y*.

For two metric spaces (X_i, d_i), $i = \overline{1,2}$, the Gromov–Hausdorff distance is then defined using the formula [29].

$$d_{GH}(X_1, X_2) = \frac{1}{2} \inf_{C \in C(X_1, X_2)} \sup_{(x_1, x_2), (y_1, y_2) \in C} | d_1(x_1, y_1) - d_2(x_2, y_2) |.$$

Another definition of the Gromov–Hausdorff distance can be given by the following formula:

$$d_{GH}(X_1, X_2) = \inf\{d_H(j_1(X_1), j_2(X_2)) \mid j_i : X_i \to \ell^\infty \text{ are isometric embeddings}\}.$$

The Gromov–Hausdorff distance was introduced by M. Gromov [30].

The Gromov–Hausdorff distance can also be defined for subspaces of the Euclidean space [31]. Let Iso ($\mathbb{R}^n$) denote the isometry group of the Euclidean space $\mathbb{R}^n$. Then the Gromov–Hausdorff distance is given by the formula:

$$d_{H,\mathrm{Iso}}(A, B) = \inf\left\{d_H(A, T(B)) \mid T \in \mathrm{Iso}(\mathrm{R}^n)\right\}.$$

An estimate for non-empty compact subsets in an *n*-dimensional Euclidean space $\mathbb{R}^n$ was provided in [31]:

$$d_{GH}(A, B) \le d_{H,\mathrm{iso}}(A, B) \le c'_n (\max\{\mathrm{diam}\, A, \mathrm{diam}\, B\} d_{GH}(A, B))^{1/2},$$

where c'_n is a constant depending only on n.

The following estimate was given in [31]:

$$d_{H,\mathrm{iso}}(A, B) \le \frac{5}{4} d_{GH}(A, B),$$

for any non-empty compact subsets $A, B \subset \mathrm{R}$.

Fréchet Distance

Let (*X*, *d*) be a metric space. A curve in a metric space *X* is an embedding γ: [0, 1]→*X*. We identify γ with γα for every homeomorphism α: [0, 1]→[0, 1]. By *C*(*X*) we denote the set of all curves in *X*.

The Fréchet distance between two curves γ_1, γ_2: [0, 1]→*X* is defined as

$$d_F(\gamma_1, \gamma_2) = \inf\{\sup\{d(\gamma_1(\alpha(t)), \gamma_2(t)) \mid t \in [0,1]\} \mid \alpha : [0,1] \to [0,1] \text{ is an increasing homeomorphism}\}.$$

If the homeomorphism does not preserve orientation, we obtain the non-oriented Fréchet distance.

The discrete Fréchet distance was introduced in [32]. This distance has found many applications [34].

The discrete Fréchet distance is closely related to the classical Fréchet distance between polygonal curves in Euclidean spaces.

A polygonal curve is a mapping *f*: [0, 1]→$\mathbb{R}^k$ such that the interval [0, 1] is partitioned by points $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$, and the function *f* is linear on each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$. That is,

$$f(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} f(t_i) + \frac{t - t_i}{t_{i+1} - t_i} f(t_{i+1}), t \in [t_i, t_{i+1}].$$

Let *g*: [0, 1]→ $\mathbb{R}^k$ be another polygonal curve with the corresponding partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{m-1} < \tau_m = 1$. A coupling of functions *f* and *g* is a sequence of pairs of non-negative integers:

$$L = ((0,0) = (a_1, b_1), \ldots, (a_{q-1}, b_{q-1}), (a_q, b_q)),$$

such that $a_{i+1} = a_i$ or $a_{i+1} = a_i + 1$, and $b_{i+1} = b_i$ or $b_{i+1} = b_i + 1$, with $a_q \ge n$, $b_q \ge m$. The length of *L* is the number

$$\|L\| = \max_{i=1,\ldots,q} d(f(t_{a_i}), g(\tau_{b_i})).$$

The discrete Fréchet distance between polygonal curves *f* and *g* is then defined as

$$d_{dF}(f, g) = \min\{\|L\| \mid L \text{ is a coupling of } f \text{ and } g\}.$$

It has been proven that the computational complexity of d_{dF} is $O(nm)$.

In [34] it was shown that there exist algorithms for computing the continuous Fréchet distance in time $O(mn\log^2(mn))$, where m, n the numbers of segments in the curves.

There also exists a non-monotonic Fréchet distance [34, 37]. The comparison of polygonal curves and their similarity in the Fréchet metric are presented in [36, 37].

Convergence of a sequence of metric curves with respect to the Hausdorff distance does not imply its convergence with respect to the Fréchet distance.

Gromov–Fréchet Distance Between Curves

This metric was first introduced in [38].

Let γ_1, γ_2 be two metric curves. We define the Gromov–Fréchet distance $d_{GF}(\gamma_1,\gamma_2)$ as follows:

$$d_{GF}(\gamma_1,\gamma_2) = \inf\{d_F(i(S), j(T)) \mid i: S \to Z,\ j: T \to Z \text{ are embeddings into a metric space } Z\}.$$

In [38] it was proved that the Gromov–Fréchet distance between metric curves is bounded below by the Gromov–Hausdorff distance between their supports.

The discrete Fréchet distance was introduced in [39, 40]. To define the discrete Gromov–Fréchet distance, we assume that the curves considered are polygonal in the sense that they are isometric to polygonal curves in normed spaces.

Analogously to the Gromov–Hausdorff distance, we can consider the following notion. Suppose that the curves γ_1, γ_2 are curves in the space R^n. Then the Gromov–Fréchet distance is defined as:

$$d_{F,\mathrm{iso}}(\gamma_1,\gamma_2) = \inf\{d_F(\gamma_1, T(\gamma_2)) \mid T \in \mathrm{Iso}(\mathrm{R}^n)\}.$$

Obviously, $d_{GF} \ge d_{F,\mathrm{iso}}$.

Algorithms for computing distances between polygons based on metrics

We describe algorithms for finding the distance between polygons based on the analyzed metrics.

Algorithm for computing the distance between polygonal curves based on the Fréchet metric

In [32], an algorithm for computing the discrete Fréchet distance was developed.

We detail the algorithm for computing the discrete Fréchet distance for two given polygonal curves. In our case, polygonal curves are the contours of polygons C_1 and C_2. In the presented algorithm, these curves are denoted by P and Q.

Let $\sigma(P) = (u_1, u_2, ..., u_p)$, $\sigma(Q) = (v_1, v_2, ..., v_q)$ be ordered sets of points of the respective curves. A coupling L between P and Q is a sequence of pairs:

$$L = \left(\left(u_{a_1}, v_{b_1}\right), \left(u_{a_2}, v_{b_2}\right), \ldots, \left(u_{a_m}, v_{b_m}\right)\right),$$

such that:

$$a_1 = b_1 = 1,\ a_m = p,\ b_m = q,$$

and at each step:

$$(a_{i+1} = a_i \text{ or } a_i + 1),\ (b_{i+1} = b_i \text{ or } b_i + 1).$$

That is, one may either stay at the same element or move to the next one.

This models the allowed movement while traversing the curves – one step forward, without jumps backward.

The length of the coupling sequence L is denoted by ||L|| and is defined by the formula:

$$\|L\| = \max_{i,j=1,\ldots,m} d\left(u_{a_i}, v_{b_j}\right),$$

where $d\left(u_{a_i}, v_{b_j}\right)$ is the distance between the points u_{a_i} and v_{b_j}.

The distance $d\left(u_{a_i}, v_{b_j}\right)$ is determined according to the following conditions:

if i = 1 and j = 1, then this distance is computed as the Euclidean distance between the points: $dE = \sqrt{(v_{b_1} - u_{a_1})^2}$;

if i > 1 and j = 1, then the distance is computed as: $\max\left\{d\left(u_{a_{i-1}}, v_1\right), dE\left(u_{a_i}, v_1\right)\right\}$;

if i = 1 and j > 1, then the distance is computed as: $\max\left\{d\left(u_1, v_{b_{j-1}}\right), dE\left(u_1, v_{b_j}\right)\right\}$;

if i > 1 and j > 1, then the distance is computed as:

$$\max\left\{\min\left(d\left(u_{a_{i-1}}, v_{b_j}\right), d\left(u_{a_{i-1}}, v_{b_{j-1}}\right), d\left(u_{a_i}, v_{b_{j-1}}\right)\right), dE\left(u_1, v_{b_j}\right)\right\}.$$

Thus, we seek the maximum among the distances between the first point of curve *P* and all points of curve *Q*.

The Fréchet distance in this case is computed as follows:

$$d_F(P,Q) = \min_{j=1,\dots,m} \|L\|_j .$$

The idea is as follows:

– To compute the Fréchet distance, we consider all possible couplings (paths) between the two curves.

– We then choose the one for which the maximum distance in the pairs is minimal (we minimize the maximum step).

– This is an analogue of the min–max approach: minimizing the maximum.

Thus, the obtained minimum of all maximum distances will be the distance between the curves in the Fréchet metric.

Below we present pseudocode for this algorithm.

```
Function dF(P,Q): real;
input: polygonal curves P = (u1, ..., up) and Q = (v1, ..., vq).
return: _dF (P,Q)
ca : array [1..p, 1..q] of real;
function c(i, j): real;
begin
if ca(i, j) > −1 then return ca(i, j)
elsif i = 1 and j = 1 then ca(i, j) := d(u1, v1)
elsif i > 1 and j = 1 then ca(i, j) := max{ c(i − 1, 1), d(ui, v1) }
elsif i = 1 and j > 1 then ca(i, j) := max{ c(1, j − 1), d(u1, vj) }
elsif i > 1 and j > 1 then ca(i, j) :=
max{min(c(i − 1, j), c(i − 1, j − 1), c(i, j − 1)), d(ui, vj) }
else ca(i, j) = 1
return ca(i, j);
end; /* function c */
begin
for i = 1 to p do for j = 1 to q do ca(i, j) := −1.0;
return c(p, q);
end.
```

Algorithm for Computing the Distance Between Polygons Based on the Discrete Hausdorff Metric

The computation of the discrete Hausdorff distance between regions will be carried out for the case of convex regions.

Let us consider convex regions O_1, $O_2 \subseteq \mathbb{R}^2$. It is known that the Hausdorff distance between convex regions O_1 and O_2 is equal to the Fréchet distance between their boundaries ∂O_1, ∂O_2 [41].

Therefore, to determine the discrete Hausdorff distance, we will use the algorithm for computing the discrete Fréchet distance between the contours of images.

The pseudocode of the algorithm is as follows:

```
Function dH(P,Q): real;
input: polygonal curves P = (u1, ..., up) and Q = (v1, ..., vq).
return: dH (P,Q)
begin
Function dF(P,Q): real;
dH (P,Q) = dF(P,Q);
end.
```

Algorithm for Performing Isometric Transformations on Polygons

To determine the minimal distance between the regions O_1 and O_2, it is necessary to perform isometric transformations. The isometric transformations must ensure the maximum intersection, i.e., S = O1 ∩ O2 → max. For polygonal region O_1, we construct the set of chords $\{h_1, h_2, \dots, h_k\}$, and for polygonal region O_2 – the set of chords $\{l_1, l_2, \dots, l_p\}$.

Based on the algorithm [42], we find the set of weighted chords for polygonal region O_1: $\left\{h_{w_1}, h_{w_2}, \dots, h_{w_k}\right\}$, O_2: $\left\{l_{w_1}, l_{w_2}, \dots, l_{w_p}\right\}$.

We compute the centers of mass $M_1(x_{O_1}, y_{O_1})$ and $M_2(x_{O_2}, y_{O_2})$ for the regions O_1 and O_2. Next, we perform a parallel translation T and rotation R of region O_2 to O_1 to maximize their intersection.

The pseudocode of the algorithm is presented below (IT – isometric transformations).

```
Function IT (P,Q): real;
input: polygonal curves P = (u1, ..., up) and Q = (v1, ..., vq).
return:  S (P,Q)
begin
Finding sets of weighted chords O1 – {h_{w_1}, h_{w_2}, ..., h_{w_k}}, O2 – {l_{w_1}, l_{w_2}, ..., l_{w_p}}
Finding the center of mass of polygons M_1(x_{O_1}, y_{O_1}) and M_2(x_{O_2}, y_{O_2})
Realization isometric transformations of polygons T and R.
end.
```

The algorithm for implementing the metric d_{GF} is a combination of the algorithm for computing the distance between polygonal curves based on the discrete Fréchet metric and the algorithm of isometric transformations of polygons.

The algorithm for implementing the metric d_{GH} is similarly a combination of the algorithm for computing the distance between polygonal curves based on the discrete Fréchet metric and the algorithm of isometric transformations of polygons.

Computer Experiments

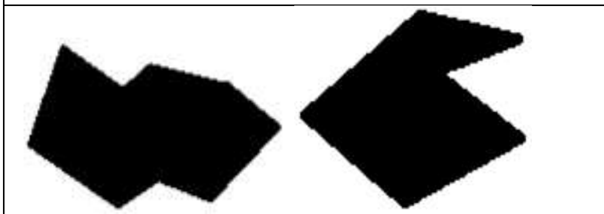
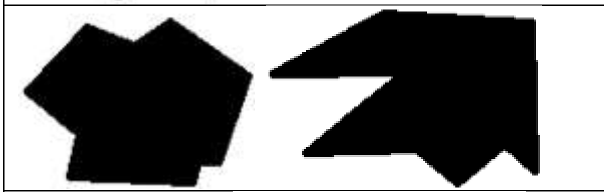
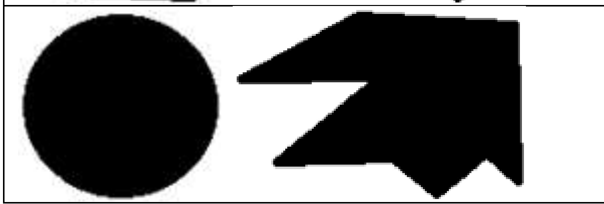
Computer experiments were carried out using the example of polygons (Table 1).

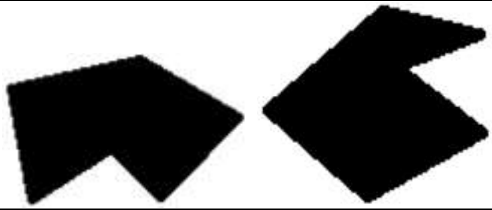
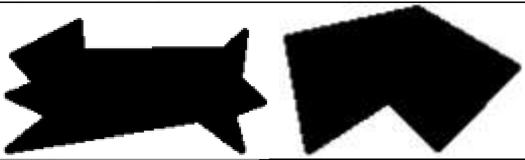
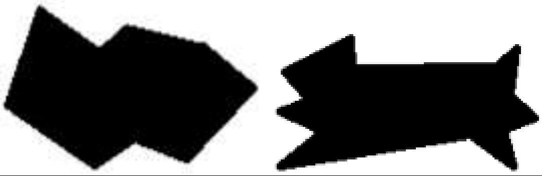
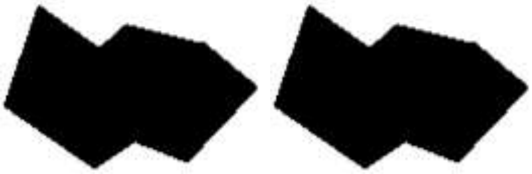
The dataset consists of a set of images with a size of 174 × 120 pixels in grayscale. In the processing stage, threshold segmentation is applied to separate the background from the object. At the next stage, contour extraction is performed.

The software module was implemented in the Java programming language in combination with the OpenCV library. Minimum system requirements: RAM – 1 GB, HDD – 100 GB. Operating system: Windows, Linux, MacOS. Programming language: Java8+, OpenCV 3.0, IDE: IntelliJ IDEA.

Table

Results of computational experiments with metrics

Image pair	**Fréchet distance**	**Hausdorff Distance**	**Gromov–Fréchet distance**	**Gromov–Hausdorff Distance**
	64.14	27.07	54.58	26.9
	35.44	35.12	35.12	35.11
	55.0	55.0	46.4	46.4

	36.76	35.46	36.06	36.12
	95.21	55.5	66.7	54.81
	64.1	27.07	54.58	26.90
	0	0	0	0

Conclusions

In the article, classical and modern metrics used for image comparison are analyzed.

The Fréchet distance characterizes the closeness of polygons based on the analysis of curves (contours). The smaller it is, the more similar the contours are. The Hausdorff distance characterizes the similarity of polygonal regions bounded by contours. This distance is smaller when the polygonal regions are more similar.

The Gromov–Hausdorff and Gromov–Fréchet distances show the minimal distance between regions and polygonal contours, respectively.

This is confirmed by computational experiments in Table.

The article presents algorithms for computing distances between polygons based on metrics that have low computational complexity.

The limitations of the Fréchet, Hausdorff, Gromov–Fréchet, and Gromov–Hausdorff metrics consist in the fact that they compute pointwise distances between polygons. In practice, it is also necessary to compute distances between images that lie within a range.

Therefore, further research will focus on the development of a fuzzy Fréchet metric, a fuzzy discrete Fréchet metric, and a fuzzy combined discrete Fréchet and Hausdorff metric.

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Mykola Berezkyi **Микола Березький**	Graduate student of the Department of Computer Engineering West Ukrainian National University https://orcid.org/0000-0001-6507-9117 e-mail: mykolaberezkyy@gmail.com	Аспірант кафедри комп'ютерної інженерії, Західноукраїнського національного університету